Diagonal Proof, Classical Mathematics and Intuitionism

Clement Manuelpillai

1 Introduction

The controversial topic of Mengenlehre (the theory of infinite sets or set theory) was created by Georg Cantor and it is claimed that the Diagonal Pro= of Method (henceforth, *DPM*) shows that there are different infinities [see [2]] DPM and the result regarding infinities was well accepted by David Hilbert and a mathematics with an ideology and orthodoxy was developed with set the ory as the foundation of all branches of mathematics This mathematics s is called classical mathematics at this time and David Hilbert is generally regarded as the leader of this school of mathematics. Classical mathematics had its own setbacks with paradoxes and DPM was under suspect and a rival school of mathematics emerged. The founder of this school of mathematics is LEJ Brouwer who rejected DPM and this mathematics is called intuitionism. Intuitionism has its own distinct ideology and orthodoxy that is in sharp conflict with classical mathematics where all branches of mathematics are developed independently with foundations that are not based on set theory [see [1, 3] and the references for intuitionism cited in [3]].

DPM played a crucial role in the development of both classical mathematics and intuitionism. We find that both classical mathematics and intuitionism are flawed in their respective foundations. In this paper, we point out the flaws in their respective foundations and propose changes to set the foundations straight.

In the case of classical mathematics, it not the acceptance of DPM but rather the acceptance of the claim that there are different infinities is flawed This is studied in section 2. The construction of real numbers plays a central role in mathematics. We find that the construction of real numbers in classical mathematics is unacceptable and this is shown in section 3. We also provide a construction of real numbers for classical mathematics in this section. The changes we propose for classical mathematics are the following and it is studied in section 5.

- Remove set theory from classical mathematics.
- Keep the common language meaning of infinity and the mathematical = infinity the same.

- Construct real numbers as in section 3.

In the case of intuitionism, as it stands, is flawed in its foundations and this is due to the rejection of DPM. This is studied in section 4. In classical mathematics, the construction of real numbers in intuitionism is also unacceptable and this is shown in section 3. We also provide a construction of real numbers for intuitionism in this section. Here, by using a diagonal process, we construct the species of irrational numbers= in [0,1] as a fan. This construction is carried out without the standard use of choices that is employed in the spread law in the construction of $[0, 1]$ as a fan [see [1]]. The changes we propose for intuitionism are the following lowing and it is studied in section 4.

- Remove the notion of choices in defining a sequence.
- Incorporate DPM into the proof method of intuitionism.
- Construct only the irrational numbers as a fan or a spread.

Before we proceed, we defend the philosophy of intuitionism in response to Snapper's remarks about intuitionism in [3]. Snapper presents an in depth analysis of classical mathematics and intuitionism in [3]. He points out that his analysis of the proof method of intuitionism applies just as well to any other constructive mathematics and concludes that it is an error in philosophy to define truth and falsity as in intuitionism. His position is hat the law of the excluded middle (*LEM*) holds for truth and that it should not be rejected in mathematics. We argue to the effect that classical mathematics, intuitionism or any other constructive mathematics are all free of errors in their respective philosophies.

Classical mathematics is based on truth and truth determines proof of methods On the other hand, intuitionism is based on proof method and proof method determines truth. That is we have two opposing philosophies of mathematics. One may disagree with a certain philosophy but that does not mean that there are errors in that philosophy. The basic question is that whether LEM has to hold for truth in mathematics. LEM is part of logic. Is mathematics part of logic? Poincare and Snapper are among those who insist that mathematics is not part of logic. If mathematics is not part of logic then we have a choice of logic in mathematics. Although LEM is rejected in intuitionism, the law $\sigma \to \sigma$ is false is accepted in intuitionism and it is used in proofs

by contradiction to seek truth. This law is accepted because of the meaning ascribed to the word 'not' and more importantly, to avoid contradictions in mathematics. The philosophy of intuitionism is to seek truth. This forces Brouwer to reject LEM and replace it by the well known intuitionistic analogue that requires a proof in the use of LEM.

2 The Diagonal Proof Method

Wittgenstein and Brouwer are among those who do not accept DPM. We outline DPM

Theorem 2.1 $[0, 1]$ is uncountable.

Proof: Assume that [0,1] is denumerable. Hence there exists an enumeration of [0,1]. Hence, every real number in [0,1] belongs to the enumeration. Hence, the diagonal sequence belongs to the enumeration (p). However, the diagonal process sequence $\neq$ every member of the enumeration (?). Hence, the diagonal process sequence does not belong to the enumeration ($\neq p$). Hence, [0,1] is non-denumerable.
A proof method is accepted or rejected based on reasoning. It is easy to see from the outline that the questionable implication is (?). If we accept (?) then we accept DPM. It is therefore safe to say that Wittgenstein and Brouwer reject (?). Hodges states in [2] that Wittgenstein seems to single out the diagonal process when it involves the entire set [0,1]. That is, the diagonal process cannot construct a sequence that is different from all the sequences. Wittgenstein is placing a limitation on the constructive power of the diagonal process. That is, if a set already has all the sequences then the diagonal process cannot construct a different sequence. On the other hand, Brouwer rejects (?) because the enumeration of [0,1] is an infinite set. However, Brouwer would accept (?) if the enumeration were a finite set. So, when does the results for the finite case extend to the infinite? There are instances where the results for the finite extend to the infinite while there are instances where it does not. So the question is whether the result for the finite extends to the infinite in this case. We show that the result does extend to the infinite and also refute Wittgenstein's objection to (?).

Example 2.1 Conisder the set $I = \{\frac{1}{2^n}, n = 1, 2, 3, \ldots\}$. In classical mathematics or in intuitionism, the number 0 does not belong to I. We enumerate the set I in base 2.

$$\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 1 & 0 & 0 & \ldots \end{array}$$

The diagonal sequence is the unit sequence and the diagonal process sequence is the zero sequence. The diagonal process constructs the zero sequence that does not belong to I. Similarly in the outline of DPM, the diagonal process does construct a sequence that does not belong to the given denumerable set of real numbers. Regardless of whether we have a constructed denumerable set or a set that is assumed to be denumerable, the diagonal process does construct a sequence that does not belong to the given set. The only difference is that we do construct a sequence from a constructed denumerable set, where as we deduce a construction of a sequence from a set that is assumed denumerable. We view the diagonal process from a different perspective

Example 2: Sequences of 0's and 1's having n terms We cannot enumerate all these sequences with only n sequences because there are 2^n sequences of 0's and 1's. We may also show by a DPM that an enumeration with n sequences does not exist. Since the terms of an infinite sequence can be changed just the same way as the terms of a finite sequence this result holds in the infinite case.

The examples 2.1 and 2.2 should convince the reader that the arguments of Wittgenstein and Brouwer are untenable. In classical mathematics, since there does not exist a bijection between the natural numbers and [0,1], [0,1] is claimed to be bigger in size than that of the natural numbers. The idea is that there are different infinities. The size of [0,1] is of the size of the set of infinite sequences of 0's and 1's. As n goes to infinity, 2^n goes to infinity. Notice that, as n goes to infinity, 2^n keeps taking larger and larger finite values and 2^n remains finite. Therefore, we do not have a different size of infinity for $[0, 1]$ and we reject the notion of transfinite cardinals.

We accept that [0,1] is non-denumerable. However, just because [0,1] is non-denumerable, that does not mean that it is bigger in size than the natural numbers. We challenge the validity of the definition that [0,1] is non- denumerable means that it is bigger in size than the natural numbers. Let us defend our position from a different perspective. Recall that every finite sequence of 0's and 1's corresponds to a

rational number. However, every infinite sequence of 0's and 1's does not correspond to a rational number since there are non-terminating, non-repeating infinite sequences of 0's and 1's. This is an example of a result that holds for finite sequences while it does not extend to the infinite sequences. These non-terminating, non-repeating sequences correspond to irrational numbers. An irrational number is negatively defined as a real number that is not a rational number.

Theorem 2.2 $\sqrt{2}$ is an irrational number.

Proof (Outline): Assume that $\sqrt{2}$ is a rational number. So, $\sqrt{2} = \frac{a}{b}$, with $(a, b) = 1$, since every rational number is reducible to lowest terms (p). o a, b are even. Hence, $\sqrt{2}$ is not in lowest terms ($\neq p$). Hence, $\sqrt{2}$ is an irrational number.

The assumption that $\sqrt{2}$ is reducible leads to the contradiction that it is irreducible. The correct deduction is that $\sqrt{2}$ is not a rational number. We cannot deduce that it is a special kind of a rational number that is an irreducible rational number. Compare this with Theorem 2.1. The assumption that [0,1] is denumerable leads to the contradiction that $[0, 1]$ is non-denumerable the construction of a number by the diagonal process that does not belong to [0,1]. The correct deduction is that [0,1] is non-denumerable. We cannot deduce that [0,1] is a special kind of an infinite set that is bigger in size than the natural numbers. We could define irrational numbers as irreducible rational numbers and proceed to introduce a theory of irreducible rational numbers. This theory as well as the theory of infinite sets is flawed in the foundations. There is only one kind of rational numbers and there is only one kind of infinite sets. The theory of infinite sets is a mistake.

3 Construction of Real Numbers

In intuitionism, a sequence is defined by a law or by choices or by using both. In this section, we restrict the discussion of the sequences of intuitionism to only those defined by a law. In classical mathematics, a sequence is defined by a law. The notion of a sequence in classical mathematics and the notion of a sequence defined by a law in intuitionism are different. This difference is due to what we mean by existence. Existence in classical mathematics may not require a computation while existence in intuitionism requires it. There are examples of sequences of classical mathematics that are not sequences of intuitionism (see

[1]). Further more, in intuitionism, only a finite segment of a sequence is known, while in classical mathematics the entire sequence is known. For example, consider the zero sequence. In intuitionism, this is an infinitely proceeding sequence (ips) in which only a finite segment is known. We need to compute the terms of the sequence one at a time and the construction of the sequence is incomplete. In classical mathematics, this sequence is an ips where the terms of the sequence are defined for every natural number and therefore the infinite sequence is known in its entirety. That is, the construction of the sequence is completed. This does not mean that the sequence is a completed set. Since the notion of a sequence in classical mathematics is different from that in intuitionism, the Cauchy sequences are also different.= Since a real number is defined as a Cauchy sequence of rational numbers, the classical continuum is different from the intuitionistic continuum.

In both classical mathematics and intuitionism, when real numbers are constructed, the construction is carried out with rational numbers. In this process, rational numbers that have a construction by the operation called division are constructed again. That is, one constructs the constructed rational numbers from the constructed rational numbers. What kind of a construction is this? Surprisingly, these constructions are accepted as legitimate constructions. To make matters worse, in classical mathematics, the irrational numbers are not even constructed. They are simply collected as Wittgenstein had noticed. We therefore, allow only the construction of irrational numbers using rational numbers.

3.1 Construction of Irrational Numbers in [0,1] for Intuitionism

We work in base 2. In order to carry out the diagonal process, we rep resent every terminating binary expansion of a rational number in [0,1] as a repeating binary expansion in 0. Let A denote the species consisting of terminating and eventually periodic binary expansions of rational numbers in [0,1] except the terminating expansion of the number 1. For every rational number in [0,1] other than 1 with double representation, both expansions are included in A. It is clear that the species A is denumerable. Let $a_1, a_2, \ldots$ designate an enumeration of A. Let $a_n(i)$ denote the ith digit of the binary expansion of a_n. We construct a binary expansion b such that $b(i) \neq a_i(i), i = 1, 2, \ldots$.

Here $b(i)$ denotes the ith digit of b. Thus the ith digi= t of b is never equal to the ith digit of the ith binary expansion a_i. Since b differs from every element of A, b is an irrational number in [0,1]. The diagonal process on every enumeration of A constructs irrational numbers in [0,1]. We use this diagonal process to construct the irrational numbers in [0,1] as a fan.

Spread Law: Let $a_1, a_2, \ldots$ designate an enumeration of the species A. Admissible sequences consist of 0's and 1's. Let $a_n(i)$ denote the ith digit of the element a_n of this enumeration of the species A, and let $b_n(i)$ equal 1 if $a_n(i) = 0$, and 0 if $a_n(i) = 1$. Every $b_n(1)$ is a one-member admissible seqeunce. If $b_n^1(1), b_n^2(2), \ldots, b_n^k(k)$ is an admissible sequence, then $b_n^1(1), b_n^2(2), \ldots, b_n^{k+1}(k+1)$ is an admissible sequence iff $n_{k+1} \neq n_k \neq n_1$.

Complementary Law: To the sequence $b_n(1), b_n(2), \ldots, b_n(k)$ (if admissible) is assigned the rational number $\sum_{i=1}^{k} \frac{b_n^i(i)}{2^i}$. We show that the above fan coincides with species of irrational numbers in [0,1].

Let $i_g = .i_1 i_2 i_3 \ldots$ denote any irrational number in [0,1]. There are infinitely many 0's, 1's in $x = n$, for every n. Hence for every position of i_g , an infinite number of a_n's can be chosen to carry out the diagonal process. To complete the proof, we show that every element of A can be selected for the diagonal process. We show that each a_n differs from i_g at infinite number of positions. If i_g and a_n differ only at a finite number of positions, then both expansions i_g and a_n will have identical tail ends which obviously cannot be the case. Hence each a_n differs from i_g at infinite number of positions. Therefore, there exists an enumeration of A on which the diagonal process constructs i_g.

Remark 3.1 The species of irrational numbers is constructed as a spread; assign the rational number $\mathcal{Z} + \sum_{i=1}^{k} \frac{b_{n_i}(i)}{2^i}$ in the above complementary lae, where $\mathcal{Z}$ is the species of integers.

3.2 Construction of Real Numbers for Classical Mathematics

We argued that the construction of real numbers in classical mathematics ans intuitionism are both unacceptable. The construction in clasical mathematics simply collects irrational numbers. The construction of classical mathematics should not even be called a construction. However, the mathematics of intuitionism is very precise with explicit

directions in the construction of real numbers. So, we use the mathematics of intuitionism in the construction of real numbers for classical mathematics. We have explained the difference between the sequences of classical mathematics and the sequences of intuitionism that are defined by a law in the beginning of this section. The construction of irrational numbers for classical mathematics is the same as the above construction for intuitionism. The construction needs to be vi viewed from a classical mathematics standpoint to distinguish terminology.

4 Intuitionism

As mentioned in section 3, in intuitionism, a sequence is defined by a law or by choices or by using both. Choice sequences are used in the spread law in the construction of [0,1] as a fan. We stated in section 3 that the construction is unacceptable. There is also another reason why this construction is unacceptabe; the rational numbers that are complete objects are constructed as incomplete objects. We do not need choice sequences to carry out the program of intuitionism. It is also very difficult to justify the notion of choice sequences. Further more, the fundamental theorems produced from its aplications are anti-intuitive. We therefore, call for the removal of the notion of choices in defining a sequence. We also call for the incorporation of DPM into the poof method of intuitionism. This in turn, enables us to construct the irrational numbers in [0,1] as a fan as we did in section 3. Now the path is clear to work out the fundamental theorems of real analysis. This path leads to fundamental theorems of real analysis that at are different from intuitionism.

Theorem 4.1 A real valued function defined only on the irrational numbers = in [0,1] is uniformly continuous.

Proof: Since the species of irrational numbers in [0,1] coincides with the fan I, we use the fan theorem to prove this. The proof is essentially the same as that of the uniform continuity theorem found in [1] with some obvious modifications that the reader can easily see. Therefore, we omit the details.

Theorem 4.2 *The Uniform Continuity Theorem:* A point-wise continuous real valued function f defined on [0,1] is uniformly continuous.

Proof: By the previous theorem, the function f restricted to the irrational numbers is uniformly continuous. Since the irrational numbers in [0,1] is dense in [0,1] and f is point-wise continuous, it follows that f is uniformly continuous on [0,1].

It is not very difficult to formulate and prove the other fundamental theorems of real analysis. By adhering to the three changes we called for in the introduction (section 1), the mathematics of the other branches of intuitionism such as algebra, topology and so on could be worked out as well.

5 Classical Mathematics

We begin with a discussion of the theory of functions of one variable.
Example 5.1: The characteristic function f that takes the value 1 on the irrational numbers and 0 on the rational numbers. This is a well defined function in classical mathematics where as it is not well defined in intuitionism or in our intuitionism.
What is the function value for the Euler number c?

In classical mathematics, intuitionism or in our intuitionism, we look at the corresponding construction or feal numbers. In classical mathematics, since c is either rational or irrational (LEM), there exists a value. It is not necessary to know the value or the status of the Euler number

In intuitionism, from the construction of real numbers, there is an ips in= the construction corresponding to c. However, since we do not know the status of c, we cannot assign a value for c. It is required to know the status of c. Therefore, the above function is not well defined.

In our intuitionism, again, we look at the construction of real numbers. c belongs to the construction becaue of congruency. However, since we do not know the status of c, we do not know whether a rational number or an ips that corresponds to c. Hence, we cannot assign a value for c. Therefore, the above function is not well defined.

The theory of functions of one variable does deal with unsolved problems of mathematics. However, the theory does not depend on solutions to unsolved problems. In intuitionism, unsolved problems play only a secondary role in the form of Brouwerian counter examples. These unsolved problems are part of number theory and number theory is not the foundation for real analysis. Similarly, set theory should not be the foundation for real analysis.

We reject the notion of transfinite cardinals. The common language meaning of infinity and the mathematical infinity should be the same. By adhering to the three changes we called for in the introduction (section 1) and with appropriate changes to [1] the classical mathematics could be worked out without much difficulty. We also call for the removal of the popular integration theory called Lebesgue measure theory as it is founded on the notion of transfinite cardinals.

The text books of classical mathematics in the elementary courses also need revision in several areas. Needless to say that technology is a useful tool in the class room. However, it is the same dance with a different music.

6 Acknowledgement and Dedication

I am grateful to Peter Sauter, Senior IT Manager for PERIF Formwork Systems Inc. USA for allowing me to use his computer facilites and his assistance. This paper is dedicated to Professor Ernst Snapper and to Dr. S. V. Kasynathan.

References

[1] A. Heyting, Intuitionism, An Introduction, Amsterdam, 1966.

[2] W. Hodges.' An Editor Recalls Some Hopeless Papers', The Bulletin of Symbolic Logic Vol4, No. 1, March 1998, 1-16.

[3] E. Snapper: 'What is Mathematics', Monthly, Augustj-September 1979.